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THE PRICING OF QUANTO OPTIONS UNDER THE DOUBLE SQUARE ROOT SHORT RATE MODEL

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Abstract. We derive a closed-form expression for the price of a European quanto call option when both foreign and domestic interest rates follow the double square root short rate model.

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1. Introduction

A quanto is a type of financial derivative whose pay-out currency differs from the natural denomination of its underlying financial variable. A quanto option is a cash-settled, cross-currency derivative whose underlying asset has a payoff in one currency, but the payoff is converted to another currency when the option is settled. This allows investors to obtain exposure to foreign assets without the corresponding foreign exchange risk.

Pricing quanto options based on the classical Black-Scholes(1973) [1] model has a weakness of assuming both constant volatility and constant interest rates. To overcome such weakness, in valuing quanto options, it is natural to consider stochastic interest rate or stochastic volatility models. Despite its importance, not many researches have been done on finding analytic solutions of quanto option prices under stochastic interest rate or stochastic volatility models primarily due to the sophisticated stochastic processes and inability to obtain the general closed form. However, by assuming constant interest rates, Giese(2012) [2] provided a closed-form expression for the price of a quanto option in the Stein-Stein(1991) [5] stochastic volatility model, and then Lee and Lee(2014) [3] got a closed-form expression for the price of a European quanto call option in the double square root stochastic volatility model.

In this paper, we assume the double square root stochastic interest rates and a constant volatility to obtain a closed-form expression for the price of a quanto option. Indeed, we use the double square root model which was introduced by Longstaff(1989) [4] and later modified by Zhu(2000, 2010) [6], [7] to describe the stochastic processes of both domestic and foreign interest rates.

We specify dynamics of the processes of underlying asset and short rate under the quanto measure in Section 2. Then, in Section 3, we drive a closed-form expression of a quanto option price under the model specified in the previous section. We mostly use the same notations as those in [3]. Theorem 3.4 is the main result of the paper.

2. A model specification

For a dividend paying asset with the constant dividend yield q and a constant volatility σ_S, the process of the asset price S_t may be assumed to be denominated in foreign currency X and to have the following dynamics:

$$dS_t = \left(\sqrt{r_t^X} - q\right) S_t dt + \sigma_S \sqrt{r_t^X} S_t dB_t^{\mathbb{Q}^X}, \tag{1}$$

$$dr_t^X = \kappa^X \left(\theta^X - \sqrt{r_t^X}\right) dt + \xi^X \sqrt{r_t^X} dW_t^{\mathbb{Q}^X}, \tag{2}$$

$$dr_t^Y = \kappa^Y \left(\theta^Y - \sqrt{r_t^Y}\right) dt + \xi^Y \sqrt{r_t^Y} d\hat{W}_t^{\mathbb{Q}^Y}, \tag{3}$$

where $B_t^{\mathbb{Q}^X}$ and $W_t^{\mathbb{Q}^X}$ are two standard Brownian motions under the foreign risk-neutral probability measure $\mathbb{Q}^X$, and $\hat{W}_t^{\mathbb{Q}^Y}$ is a standard Brownian motions under the domestic risk-neutral probability measure $\mathbb{Q}^Y$. Also, r_t^X and r_t^Y denote the foreign and domestic interest rates, respectively, which follow stochastic short rate processes of Longstaff(1989) [4] for the asset price S_t with constant parameters κ^i, θ^i and ξ^i for each $i = X, Y$. Since there are two square root terms in above dynamics, it is referred to as the double square root process, whose basic features are described in Chapter 3.4 of [7]. To give a closed-form expression, we should add a restriction of parameters so that $4\kappa^i\theta^i = {\xi^i}^2$ for each $i = X, Y$, which is the strong condition to be able to analytically calculate some special conditional expectation including double square roots. This particular meaning is minutely explained in [4]. In addition, it is assumed that an investor's domestic currency is Y and he wishes to obtain exposure to the asset price S_t without carrying the corresponding foreign exchange risk.

Let $Z_t^{Y/X}$ denote the price of one unit of currency Y in units of currency X and $Z_t^{Y/X}$ follows the standard Black-Scholes type dynamics under $\mathbb{Q}^X$ such as

$$Z_t^{Y/X} = \left(r_t^X - r_t^Y\right) Z_t^{Y/X} dt + \sigma_{\mathrm{FX}} Z_t^{Y/X} d\hat{B}_t^{\mathbb{Q}^X},$$

where $\hat{B}_t^{\mathbb{Q}^X}$ is a standard Brownian motion under $\mathbb{Q}^X$ and σ_{FX} is the constant volatility of the foreign exchange rate $Z_t^{Y/X}$. This model allows three constant correlation ρ, ν and β satisfying

$$dB_t^{\mathbb{Q}^X} dW_t^{\mathbb{Q}^X} = \rho dt, \quad dB_t^{\mathbb{Q}^X} d\hat{B}_t^{\mathbb{Q}^X} = \nu dt, \quad dW_t^{\mathbb{Q}^X} d\hat{B}_t^{\mathbb{Q}^X} = \beta dt.$$

Using the change of measure from $\mathbb{Q}^X$ to the domestic risk-neutral probability measure $\mathbb{Q}^Y$ with the Radon-Nikodým derivative

$$\left.\frac{d\mathbb{Q}^Y}{d\mathbb{Q}^X}\right|_{\mathcal{F}_t} = e^{-\frac{1}{2}\sigma_{\mathrm{FX}}^2 t + \sigma_{\mathrm{FX}} \hat{B}_t^{\mathbb{Q}^X}},$$

the Girsanov's theorem implies that the new processes $B_t^{\mathbb{Q}^Y}$, $W_t^{\mathbb{Q}^Y}$ and $\hat{B}_t^{\mathbb{Q}^Y}$ defined by

$$\begin{aligned} dB_t^{\mathbb{Q}^Y} &= dB_t^{\mathbb{Q}^X} - \nu\sigma_{\mathrm{FX}}dt, \\ dW_t^{\mathbb{Q}^Y} &= dW_t^{\mathbb{Q}^X} - \beta\sigma_{\mathrm{FX}}dt, \\ d\hat{B}_t^{\mathbb{Q}^Y} &= d\hat{B}_t^{\mathbb{Q}^X} + \sigma_{\mathrm{FX}}dt \end{aligned}$$

are again standard Brownian motions under the domestic risk-neutral probability measure $\mathbb{Q}^Y$, so called the *quanto measure*. Thus, the foreign exchange rate $Z_t^{X/Y}$ denoting the price in foreign currency X per unit of the domestic currency Y follows

$$Z_t^{X/Y} = \left(r_t^Y - r_t^X\right) Z_t^{X/Y} dt + \sigma_{\mathrm{FX}} Z_t^{X/Y} d\hat{B}_t^{\mathbb{Q}^Y}.$$

From (1) and (2), we also obtain the following dynamics of S_t and r_t^X under the quanto measure $\mathbb{Q}^Y$:

$$dS_t = \left\{(1+\nu\sigma_{\mathrm{FX}}\sigma_S)\sqrt{r_t^X} - q\right\} S_t dt + \sigma_S \sqrt{r_t^X} S_t dB_t^{\mathbb{Q}^Y}, \tag{4}$$

$$dr_t^X = \hat{\kappa}^X\left(\hat{\theta}^X - \sqrt{r_t^X}\right)dt + \xi^X\sqrt{r_t^X}dW_t^{\mathbb{Q}^Y} \tag{5}$$

with $\hat{\kappa}^X = \kappa^X - \beta\sigma_{\mathrm{FX}}\xi^X$ and $\hat{\theta}^X = \frac{\kappa^X\theta^X}{\kappa^X - \beta\sigma_{\mathrm{FX}}\xi^X}$. We notice that (5) maintains the same form as (2) so that $4\hat{\kappa}^X\hat{\theta}^X = {\xi^X}^2$ also has to be satisfied. Furthermore, we may assume that two standard Brownian motions $W_t^{\mathbb{Q}^Y}$ and $\hat{W}_t^{\mathbb{Q}^Y}$ are independent.

3. A closed-form expression

In this section, by using the model specified in the previous section, we will drive a closed-form expression for the price of a European quanto call option. The following three lemmas are about some special conditional expectations under the quanto measure $\mathbb{Q}^Y$, all of which are important ingredients to the main result of the paper.

Lemma 3.1. *Under the assumption of (3), we get the following equality:*

$$\mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T r_u^Y du}\middle|\mathcal{F}_t\right] = e^{A(t,T)+B(t,T)r_t^Y + C(t,T)\sqrt{r_t^Y}}, \tag{6}$$

where

$$\begin{aligned} A(t,T) &= -\frac{1}{2}\ln\gamma_2 + \frac{{\kappa^Y}^2}{2{\xi^Y}^2\gamma_1}\left[\tanh\{\gamma_1(T-t)\} - (T-t)\right], \\ B(t,T) &= -\frac{2\gamma_1}{{\xi^Y}^2}\tanh\{\gamma_1(T-t)\} \end{aligned}$$

and

$$C(t,T) = \frac{4\kappa^Y\sinh^2\left\{\frac{\gamma_1(T-t)}{2}\right\}}{{\xi^Y}^2\gamma_2}$$

with

$$\gamma_1 = \frac{\sqrt{2\xi^{Y2}}}{2}, \quad \gamma_2 = \cosh\left\{\gamma_1 (T-t)\right\}.$$

Proof. Let us define

$$y^Y\left(t, r_t^Y\right) = \mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T r_u^Y du} \middle| \mathcal{F}_t \right].$$

Then according to the Feynman-Kač formula, y^Y is the solution of the following partial differential equation:

$$\frac{\xi^{Y2}}{2} r^Y \frac{\partial^2 y^Y}{\partial r^{Y2}} + \kappa^Y\left(\theta^Y - \sqrt{r^Y}\right)\frac{\partial y^Y}{\partial r^Y} - r^Y y + \frac{\partial y^Y}{\partial t} = 0$$

with the terminal condition

$$y^Y\left(T, r_T^Y\right) = 1.$$

Now, putting our solution as the following functional form:

$$y^Y\left(t, r_t^Y\right) = e^{A(t,T) + B(t,T) r_t^Y + C(t,T)\sqrt{r_t^Y}},$$

we have the following ordinary differential equations:[1]:

$$\begin{cases} \dfrac{\partial A}{\partial t} = -\dfrac{\xi^{Y2}}{4} B(t,T) - \dfrac{\xi^{Y2}}{8} C(t,T)^2 + \dfrac{\hat{\kappa}^Y}{2} C(t,T), \\ \dfrac{\partial B}{\partial t} = -\dfrac{\xi^{Y2}}{2} B(t,T)^2 + 1, \\ \dfrac{\partial C}{\partial t} = -\dfrac{\xi^{Y2}}{2} B(t,T) C(t,T) + \hat{\kappa}^Y B(t,T) \end{cases}$$

with terminal conditions

$$A(T,T) = B(T,T) = C(T,T) = 0.$$

By solving these ordinary differential equations, we complete the proof of the lemma. □

We shall denote the price at time t of a zero-coupon bond in currency Y that matures a nominal value of 1 at time T by

$$P^Y(t,T) = \mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T r_u^Y du} \middle| \mathcal{F}_t \right].$$

We call this the T-*(zero-coupon) bond in currency* Y. Now, we will compute the value of a quanto forward contract $\mathbb{E}_{\mathbb{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right]$ from the risk-neutral valuation method.

Lemma 3.2. *Under the assumptions of (4) and (5), we get the following equality:*

$$\mathbb{E}_{\mathbb{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right] = S_t e^{-q(T-t) - \frac{\rho\sigma_S}{\xi^X}\left\{ r_t^X + \hat{\kappa}^X \hat{\theta}^X (T-t) \right\}} \times \mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T \left(c_1 r_u^X + c_2\sqrt{r_u^X}\right) du + c_3 r_T^X} \middle| \mathcal{F}_t \right]$$

[1]Due to the restriction that $4\hat{\kappa}^Y \hat{\theta}^Y = \xi^{Y2}$, the coefficient of $\frac{1}{r_t^Y}$ vanishes in the calculating course.

with

$$c_1 = \frac{\rho^2\sigma_S^2}{2}, \quad c_2 = -1 - \nu\sigma_{FX}\sigma_S - \frac{\rho\sigma_S\hat{\kappa}^X}{\xi^X}, \quad c_3 = \frac{\rho\sigma_S}{\xi^X}.$$

Proof. Applying the Itô formula to (4) together with the tower property, we obtain

$$\mathbb{E}_{\mathbb{Q}^Y}\left[S_T\middle|\mathcal{F}_t\right] = S_t e^{-q(T-t)}$$

$$(7) \qquad \times\, \mathbb{E}_{\mathbb{Q}^Y}\left[e^{(1+\nu\sigma_{\mathrm{FX}}\sigma_S)\int_t^T \sqrt{r_u^X}du - \frac{\rho^2\sigma_S^2}{2}\int_t^T r_u^X du + \rho\sigma_S\int_t^T \sqrt{r_u^X}dW_u^{\mathbb{Q}^Y}} \middle| \mathcal{F}_t\right],$$

where we may write $B_t^{\mathbb{Q}^Y}$ as $B_t^{\mathbb{Q}^Y} = \rho W_t^{\mathbb{Q}^Y} + \sqrt{1-\rho^2}W_t$ with W_t being a $\mathbb{Q}^Y$-standard Brownian motion independent on $W_t^{\mathbb{Q}^Y}$. From (5), we have

$$(8) \quad \int_t^T \sqrt{r_u^X}dW_u^{\mathbb{Q}^Y} = \frac{1}{\xi^X}\left\{ r_T^X - r_t^X - \hat{\kappa}^X\hat{\theta}^X(T-t) + \hat{\kappa}^X\int_t^T \sqrt{r_u^X}du \right\}.$$

Substituting (8) for (7), we obtain

$$\mathbb{E}_{\mathbb{Q}^Y}\left[S_T\middle|\mathcal{F}_t\right] = S_t e^{-q(T-t) - \frac{\rho\sigma_S}{\xi^X}\left\{r_t^X + \hat{\kappa}^X\hat{\theta}^X(T-t)\right\}}$$
$$\times\, \mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T\left(c_1 r_u^X + c_2\sqrt{r_u^X}\right)du + c_3 r_T^X} \middle| \mathcal{F}_t\right]$$

with

$$c_1 = \frac{\rho^2\sigma_S^2}{2}, \quad c_2 = -1 - \nu\sigma_{\mathrm{FX}}\sigma_S - \frac{\rho\sigma_S\hat{\kappa}^X}{\xi^X}, \quad c_3 = \frac{\rho\sigma_S}{\xi^X}.$$

□

In addition, we need the following lemma to get the value of $\mathbb{E}_{\mathbb{Q}^Y}\left[S_T\middle|\mathcal{F}_t\right]$. Indeed, this is an identical version of Lemma 3.1 previously introduced in [3].

Lemma 3.3. *Under the assumption of (5) together with constants c_1, c_2 and c_3, we get the following equality:*

$$(9) \quad \mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T\left(c_1 r_u^X + c_2\sqrt{r_u^X}\right)du + c_3 r_T^X} \middle| \mathcal{F}_t\right] = e^{G(t,T) + H(t,T)r_t^Y + I(t,T)\sqrt{r_t^Y}},$$

where

$$G(t,T) = -\frac{1}{2}\ln\gamma_4 + \frac{\left(\gamma_3^2 - \hat{\kappa}^{X^2}\gamma_1^2\right)(T-t)}{2\xi^{X^2}\gamma_1^2} + \frac{\left(\gamma_2\gamma_3 - 2\hat{\kappa}^X\gamma_1^2\right)\gamma_3}{2\xi^{X^2}\gamma_1^4}\left(\frac{1}{\gamma_4} - 1\right)$$
$$+ \frac{\sinh\{\gamma_1(T-t)\}\left\{\hat{\kappa}^{X^2}\gamma_1^2 - \hat{\kappa}^X\gamma_2\gamma_3 - \gamma_3^2 + \frac{1}{2}\left(\frac{\gamma_2\gamma_3}{\gamma_1}\right)^2\right\}}{2\xi^{X^2}\gamma_1^3\gamma_4},$$

$$H(t,T) = -\frac{2\gamma_1}{\xi^{X^2}}\cdot\frac{2\gamma_1\sinh\{\gamma_1(T-t)\} + \gamma_2\cosh\{\gamma_1(T-t)\}}{2\gamma_1\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}}$$

and

$$I(t,T) = \frac{2\sinh\left\{\frac{\gamma_1(T-t)}{2}\right\}}{\xi^{X^2}\gamma_1\gamma_4}$$

$$\times\left[\left(\hat{\kappa}^X\gamma_2-2\gamma_3\right)\cosh\left\{\frac{\gamma_1\left(T-t\right)}{2}\right\}+\left(2\hat{\kappa}^X\gamma_1-\frac{\gamma_2\gamma_3}{\gamma_1}\right)\sinh\left\{\frac{\gamma_1\left(T-t\right)}{2}\right\}\right]$$

with

$$\gamma_1=\frac{\sqrt{2c_1{\xi^X}^2}}{2},\quad \gamma_2=-c_3{\xi^X}^2,\quad \gamma_3=\frac{c_2{\xi^X}^2}{2},$$
$$\gamma_4=\cosh\left\{\gamma_1\left(T-t\right)\right\}+\frac{\gamma_2}{2\gamma_1}\sinh\left\{\gamma_1\left(T-t\right)\right\}.$$

Proof. The proof is identical to that of Lemma 3.1 in [3]. □

Using the results obtained in Lemma 3.1, Lemma 3.2 and Lemma 3.3, we can obtain a closed-form expression for the price of a European quanto call option. The following theorem is the main result of the paper. For convenience, we here put the predetermined fixed exchange rate to 1.

Theorem 3.4. *Let us denote the log-asset price by* $x_t=\ln S_t$*. Under the assumption of (4), the price of a European quanto call option in domestic currency* Y *with foreign strike price* K *and maturity* T *is given by*

$$C_q^Y\left(t,S_t\right)=P^Y\left(t,T\right)\left\{\mathbb{E}_{\mathbb{Q}^Y}\left[S_T|\,\mathcal{F}_t\right]P_1-KP_2\right\},$$

where P_1*,* P_2 *are defined by*

$$P_j=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\mathbf{Re}\left[\frac{e^{i\phi\ln K}f_j\left(\phi\right)}{i\phi}\right]d\phi$$

for $j=1,2$*, in which*

$$f_1\left(\phi\right)=\frac{e^{(1+i\phi)\left\{x_t-\frac{\rho\sigma_S}{\xi^X}r_t^X-\left(q+\frac{\rho\sigma_S\hat{\kappa}^X\hat{\theta}^X}{\xi^X}\right)(T-t)\right\}}}{\mathbb{E}_{\mathbb{Q}^Y}\left[S_T|\,\mathcal{F}_t\right]}\times\mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T\left(m_1r_u^X+m_2\sqrt{r_u^X}\right)du+m_3r_T^X}\,\middle|\,\mathcal{F}_t\right]$$

with

$$m_1=\frac{\rho^2\sigma_S^2}{2}\left(1+i\phi\right),\quad m_2=-\left(1+i\phi\right)\left(1+\nu\sigma_{FX}\sigma_S+\frac{\rho\sigma_S\hat{\kappa}^X}{\xi^X}\right),$$
$$m_3=\frac{\rho\sigma_S}{\xi^X}\left(1+i\phi\right)$$

and

$$f_2\left(\phi\right)=\frac{e^{i\phi\left\{x_t-\frac{\rho\sigma_S}{\xi^X}r_t^X-\left(q+\frac{\rho\sigma_S\hat{\kappa}^X\hat{\theta}^X}{\xi^X}\right)(T-t)\right\}}}{\mathbb{E}_{\mathbb{Q}^Y}\left[S_T|\,\mathcal{F}_t\right]}\times\mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T\left(n_1r_u^X+n_2\sqrt{r_u^X}\right)du+n_3r_T^X}\,\middle|\,\mathcal{F}_t\right]$$

with

$$n_1=i\phi\frac{\rho^2\sigma_S^2}{2},\quad n_2=-i\phi\left(1+\nu\sigma_{FX}\sigma_S+\frac{\rho\sigma_S\hat{\kappa}^X}{\xi^X}\right),\quad n_3=i\phi\frac{\rho\sigma_S}{\xi^X}.$$

Proof. From the risk-neutral valuation method, the price $C_q^Y(t, S_t)$ of a European quanto call option in currency Y with foreign strike price K and maturity T is given by

$$C_q^Y(t, S_t) = \mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T r_u^Y du} \max(S_T - K, 0) \middle| \mathcal{F}_t \right].$$

For a new risk-neutral probability measure $\tilde{\mathbb{Q}}^Y$, the Radon-Nikodým derivative of $\tilde{\mathbb{Q}}^Y$ with respect to $\mathbb{Q}^Y$ is defined by

$$\frac{d\tilde{\mathbb{Q}}^Y}{d\mathbb{Q}^Y} = \frac{S_T}{\mathbb{E}_{\mathbb{Q}^Y}[S_T | \mathcal{F}_t]}.$$

Thus, the price of a European quanto call option can be rewritten as

$$\begin{aligned} C_q^Y(t, S_t) &= P^Y(t,T)\,\mathbb{E}_{\mathbb{Q}^Y}\left[S_T \mathbb{1}_{\{S_T > K\}} - K \mathbb{1}_{\{S_T > K\}} \middle| \mathcal{F}_t \right] \\ &= P^Y(t,T)\left\{ \mathbb{E}_{\mathbb{Q}^Y}[S_T | \mathcal{F}_t]\,\tilde{\mathbb{Q}}^Y(S_T > K) - K\mathbb{Q}^Y(S_T > K) \right\} \\ &= P^Y(t,T)\left\{ \mathbb{E}_{\mathbb{Q}^Y}[S_T | \mathcal{F}_t] P_1 - K P_2 \right\} \end{aligned}$$

with the risk-neutralized probabilities P_1 and P_2. Now, putting $x_t = \ln S_t$, the corresponding characteristic functions f_1 and f_2 can be represented as

$$\begin{aligned} f_1(\phi) &= \mathbb{E}_{\tilde{\mathbb{Q}}^Y}\left[e^{i\phi x_T} \middle| \mathcal{F}_t \right] \\ &= \frac{1}{\mathbb{E}_{\mathbb{Q}^Y}[S_T | \mathcal{F}_t]} \mathbb{E}_{\mathbb{Q}^Y}\left[e^{(1+i\phi)x_T} \middle| \mathcal{F}_t \right], \\ f_2(\phi) &= \mathbb{E}_{\mathbb{Q}^Y}\left[e^{i\phi x_T} \middle| \mathcal{F}_t \right]. \end{aligned}$$

On the other hand, applying the Itô formula to (4), we have

$$\begin{aligned} dx_t = &\left\{ (1 + \nu\sigma_{\mathrm{FX}}\sigma_S)\sqrt{r_t^X} - \frac{\sigma_S^2}{2} r_t^X - q \right\} dt + \rho\sigma_S\sqrt{r_t^X} dW_t^{\mathbb{Q}^Y} \\ &+ \sqrt{1-\rho^2}\,\sigma_S\sqrt{r_t^X} dW_t \end{aligned}$$

with W_t being a $\mathbb{Q}^Y$-standard Brownian motion independent of $W_t^{\mathbb{Q}^Y}$. From (8), we obtain

$$\begin{aligned} f_1(\phi) = &\frac{e^{(1+i\phi)\left\{ x_t - \frac{\rho\sigma_S}{\xi^X} r_t^X - \left(q + \frac{\rho\sigma_S \hat{\kappa}^X \hat{\theta}^X}{\xi^X} \right)(T-t) \right\}}}{\mathbb{E}_{\mathbb{Q}^Y}[S_T | \mathcal{F}_t]} \\ &\times \mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T \left(m_1 r_u^X + m_2\sqrt{r_u^X} \right) du + m_3 r_T^X} \middle| \mathcal{F}_t \right] \end{aligned}$$

with

$$\begin{aligned} m_1 &= \frac{\rho^2\sigma_S^2}{2}(1 + i\phi), \quad m_2 = -(1+i\phi)\left(1 + \nu\sigma_{\mathrm{FX}}\sigma_S + \frac{\rho\sigma_S\hat{\kappa}^X}{\xi^X} \right), \\ m_3 &= \frac{\rho\sigma_S}{\xi^X}(1 + i\phi). \end{aligned}$$

Similarly, we also obtain

$$f_2(\phi) = \frac{e^{i\phi\left\{ x_t - \frac{\rho\sigma_S}{\xi^X} r_t^X - \left(q + \frac{\rho\sigma_S \hat{\kappa}^X \hat{\theta}^X}{\xi^X} \right)(T-t) \right\}}}{\mathbb{E}_{\mathbb{Q}^Y}[S_T | \mathcal{F}_t]}$$

$$\times\, \mathbb{E}_{\mathbb{Q}^Y}\left[e^{-\int_t^T \left(n_1 r_u^X + n_2\sqrt{r_u^X}\right)du + n_3 r_T^X} \,\middle|\, \mathcal{F}_t \right]$$

with

$$n_1 = i\phi\frac{\rho^2\sigma_S^2}{2}, \quad n_2 = -i\phi\left(1 + \nu\sigma_{\mathrm{FX}}\sigma_S + \frac{\rho\sigma_S\hat{\kappa}^X}{\xi^X}\right), \quad n_3 = i\phi\frac{\rho\sigma_S}{\xi^X}.$$

Here, each value of the above risk-neutral expectation was obtained in previous lemmas.

By having closed-form expressions for the characteristic functions f_1 and f_2, the Fourier inversion formula allows us to compute the probabilities P_1 and P_2 as follows:

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathbf{Re}\left[\frac{e^{i\phi\ln K} f_j(\phi)}{i\phi}\right] d\phi$$

for $j = 1, 2$. □

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